UNIVERSITY OF LEICESTER

School of Engineering

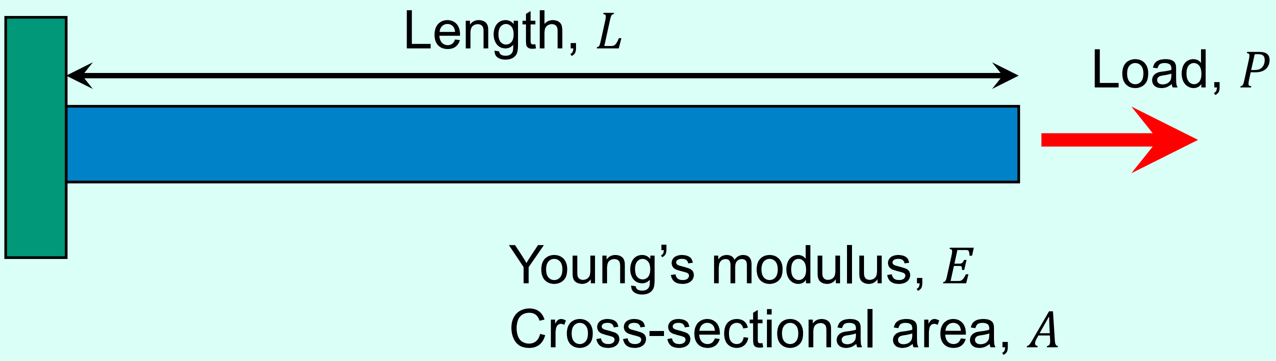
Bar Elements

Gebril El-Fallah

EG3111 – Finite Element Analysis and Design

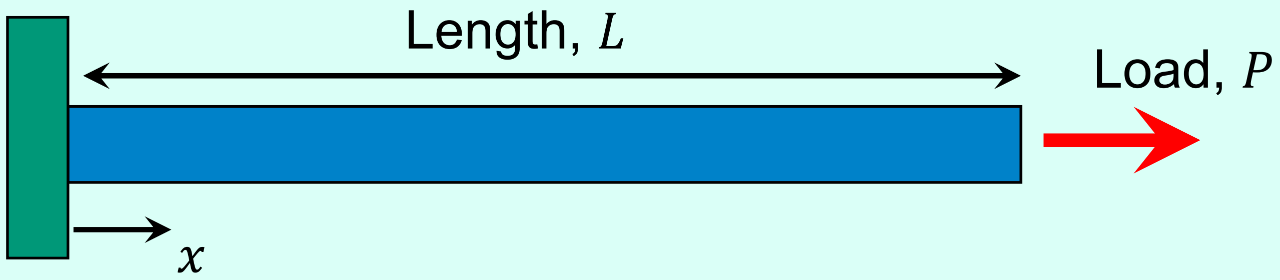
gmae2@leicester.ac.uk

3. Bar Elements

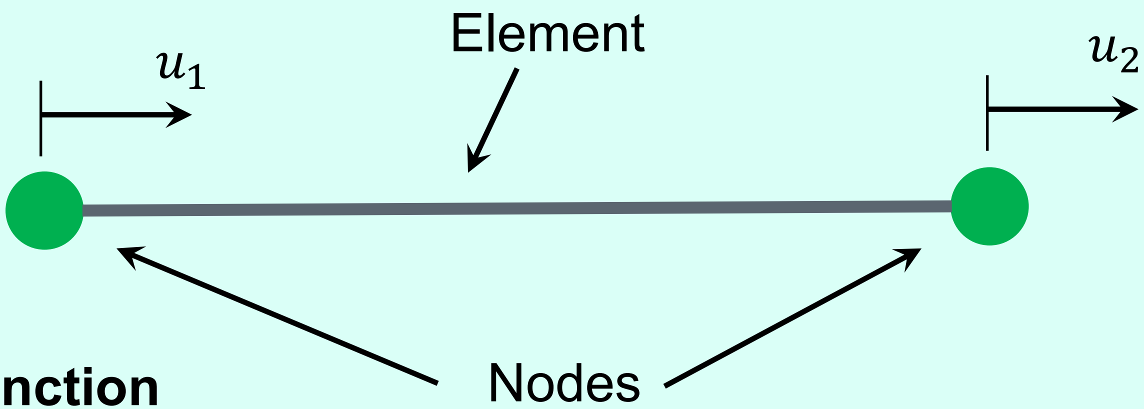


- We have already considered the problem of a bar under extension/compression in section 2.
- However, we do not wish to be formulating the total energy each time (especially for systems with many DOF) and hence wish to construct the FEM for general bar elements within a formal mathematical framework.
- This framework will also help us later on when we wish to develop other types of element, e.g beam, solid, shell.

3a. General Bar Element



Bar element



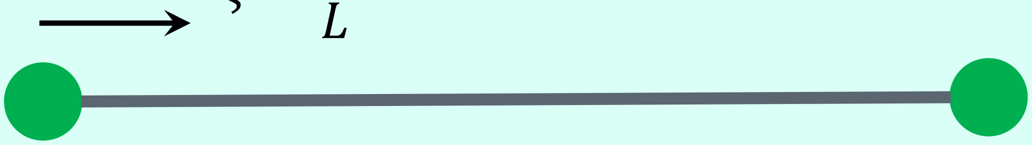
Linear shape function

$$u(x) = u_1\left(1 - \frac{x}{L}\right) + u_2\left(\frac{x}{L}\right)$$

3a. General Bar Element

Local variable, ξ

$$\xi = \frac{x}{L}$$



$\xi = 0$ $\qquad$ $\xi = 1$

$$u(\xi) = u_1(1-\xi) + u_2\xi$$

$$\Longrightarrow \quad \boxed{u(\xi) = \underline{n^e}^T . \underline{d^e}}$$

$$\underline{n^e}(\xi) = \begin{bmatrix} 1-\xi \\ \xi \end{bmatrix} \qquad \underline{d^e} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

Elemental shape function matrix

Elemental DOF matrix

3a. General Bar Element

$$\xi = \frac{x}{L} \qquad d\xi = \frac{dx}{L} \qquad \Longrightarrow \qquad dx = L d\xi$$

$\xi = 0$ $\qquad$ $\xi = 1$

$$u(\xi) = u_1(1-\xi) + u_2\xi$$

$$u(\xi) = \underline{n^{e}}^{T} . \underline{d^e}$$

Where

$$\underline{n^e}(\xi) = \begin{bmatrix} 1-\xi \\ \xi \end{bmatrix} \qquad \underline{d^e} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

Check

$$u(\xi) = \underline{n^{e}}^{T} . \underline{d^e} \qquad = [1-\xi, \xi]\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

$$\Longrightarrow \qquad u(\xi) = [(1-\xi)u_1 + \xi u_2]$$

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3a. General Bar Element

Strain-displacement matrix

$$\epsilon_x = \frac{du}{dx} = \frac{1}{L}\frac{du}{d\xi} \qquad as\ dx = Ld\xi$$

$$\epsilon_x = \frac{1}{L}\frac{du}{d\xi} = \frac{1}{L}\frac{d\left(\underline{n}^{eT}.\underline{d}^e\right)}{\mathrm{d}\xi} = \frac{1}{L}\frac{d\underline{n}^{eT}}{\mathrm{d}\xi}.\underline{d}^e = \underline{b}^{eT}.\underline{d}^e$$

Where

$$\underline{b}^{eT} = \frac{1}{L}\frac{d\underline{n}^{eT}}{\mathrm{d}\xi} = \frac{1}{L^e}\frac{d}{\mathrm{d}\xi}\begin{bmatrix}1-\xi \\ \xi\end{bmatrix} = \frac{1}{L^e}\begin{bmatrix}-1 \\ 1\end{bmatrix}$$

$$\underline{n}^e = \begin{bmatrix}1-\xi \\ \xi\end{bmatrix}$$

Elemental strain-displacement matrix
($L^e = L$ is the length of element e)

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3a. General Bar Element

Check

$$\epsilon_x = \underline{b^{e^T}}.\underline{d^e}$$

$$\epsilon_x = \frac{1}{\mathrm{L^e}}[-1,1]\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

$$\epsilon_x = \frac{1}{\mathrm{L^e}}(-u_1 + u_2)$$

$$\epsilon_x = \frac{(u_2 - u_1)}{L^e}$$

3a. General Bar Element

Elastic strain energy of an element

$$U = \frac{1}{2} EA \int_0^L \epsilon_x^2 \, dx$$

Where

$$U = \frac{1}{2} EAL \int_0^1 \epsilon_x^2 \, d\xi \qquad Eq\ 1$$

$$\underline{\epsilon} = [\epsilon_x] = \underline{b^{e^T}} . \underline{d^e}$$

$$\underline{\epsilon^T} = [\epsilon_x^T] = \underline{d^{e^T}} . \underline{b^e}$$

$$\epsilon_x^2 = \epsilon_x^T . \epsilon_x = \underline{d^{e^T}} . \underline{b^e} . \underline{b^{e^T}} . \underline{d^e}$$

Substitute ϵ_x^2 in $Eq\ 1$

$$U = \frac{1}{2} AL \int_0^1 \underline{d^{e^T}} . \underline{b^e} . E . \underline{b^{e^T}} . \underline{d^e} d\xi$$

3a. General Bar Element

Elastic strain energy of an element

$$\mathrm{U} = \frac{1}{2} AL \int_0^1 \underline{d}^{eT} . \underline{b}^{e} . E . \underline{b}^{eT} . \underline{d}^{e} d\xi$$

$$\mathrm{U} = \frac{1}{2} . \underline{d}^{eT} \int_0^1 \left\{ \underline{b}^{e} . EAL . \underline{b}^{eT} d\xi \right\} . \underline{d}^{e}$$

$$\mathrm{U} = \frac{1}{2} . \underline{d}^{eT} [k^{e}] . \underline{d}^{e}$$

3a. General Bar Element

Elastic strain energy of an element

$$\underline{\sigma} = [\sigma_x] = E.\underline{\epsilon}$$

$$\underline{\sigma^T} = \underline{d^{e^T}}.\underline{b^e}.E$$

$$\underline{\sigma^T}.\underline{\epsilon} = \underline{d^{e^T}}.\underline{b^e}.E.\underline{b^{e^T}}.\underline{d^e}$$

$$U^e = \frac{1}{2}\int_{V^e} \underline{\sigma^T}.\underline{\epsilon}\ dV = \frac{1}{2}\underline{d^{e^T}}.[k^e].\underline{d^e}$$

Where

$$[k^e] = \int_{V^e} \underline{b^e}.E.\underline{b^{e^T}}\ dV = EAL\int_0^1 \underline{b^e}.\underline{b^{e^T}}\,d\xi \qquad Eq\ 2$$

Elemental stiffness matrix

3a. General Bar Element

Check

$$\underline{b^e} = \frac{1}{L}\begin{bmatrix}-1\\1\end{bmatrix} \qquad \underline{b^{eT}} = \frac{1}{L}[-1, 1]$$

So

$$\underline{b^e}.\underline{b^{eT}} = \frac{1}{L}\begin{bmatrix}-1\\1\end{bmatrix}.\frac{1}{L}[-1, 1]$$

$$\underline{b^e}.\underline{b^{eT}} = \frac{1}{L^2}\begin{bmatrix}1 & -1\\-1 & 1\end{bmatrix}$$

Substitute $\underline{b^e}.\underline{b^{eT}}$ in Eq 2

$$[k^e] = \int_{V^e} \underline{b^e}.E.\underline{b^{eT}}\, dV = EAL\int_0^1 \underline{b^e}.\underline{b^{eT}}\, d\xi$$

$$= EAL\frac{1}{L^2}\begin{bmatrix}1 & -1\\-1 & 1\end{bmatrix}\int_0^1 d\xi = \frac{EA}{L}\begin{bmatrix}1 & -1\\-1 & 1\end{bmatrix}$$

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3a. General Bar Element

Total strain energy

Elemental strain energy

$$U^e = \frac{1}{2}\underline{d^{e^T}}.[k^e].\underline{d^e}$$

$$U = \frac{1}{2}[u_1 - u_2].\frac{EA}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

$$U = \frac{1}{2}\frac{EA}{L}.(u_1^2 - u_2 u_1 - u_1 u_2 + u_2^2)$$

$$U = \frac{1}{2}\frac{EA}{L}(u_2 - u_1)^2$$

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3a. General Bar Element

Total strain energy

Elemental strain energy

$$U^e = \frac{1}{2}\underline{d^{e^T}}.[k^e].\underline{d^e}$$

Total strain energy

$$U = \sum_{all\ elements\ e} U^e = \frac{1}{2}\underline{d^T}.[K].\underline{d}$$

The global stiffness matrix $[K]$ is the assembly of the elemental stiffness matrices $[k^e]$.

The global DOF matrix $\underline{d}$ contains all the DOF across all elemental DOF $\underline{d^e}$.

3a. General Bar Element

Finite Element Method

Potential energy of the applied loads

$$\Omega = -\underline{d}^T.\underline{f}$$

The global force matrix $\underline{f}$ contains all the forces acting of the respective DOF

Total energy

$$\Pi = \frac{1}{2}\underline{d}^T.[K].\underline{d} - \underline{d}^T.\underline{f}$$

A quadratic function of $\underline{d}$

Minimise the total energy with respect to the DOF to find the unknowns $\underline{d}$.

$$\frac{\partial \Pi}{\partial \underline{d}} = [K].\underline{d} - \underline{f} = \underline{0} \quad \Longrightarrow \quad [K].\underline{d} = \underline{f}$$

Solution

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3a. FEM Summary

Finite Element Method

To find the N degrees of freedom $\underline{d}$ solve the N simultaneous linear equations defined by

$$[K].\underline{d} = \underline{f}$$

All FEM problems can be written in this form.

Where

$[K]$ is the assembly of the $[k^e]$

$\underline{f}$ are the forces acting on each DOF

Elemental Matrices

Defined by assumed displacement field.

In 1D

$$u(x) = \underline{n}^{eT}.\underline{d}^e$$

$$\underline{b}^{eT} = \frac{1}{L}\frac{d\underline{n}^{eT}}{\mathrm{d}\xi}$$

$$[k^e] = \int_{V^e} \underline{b}^e.E.\underline{b}^{eT}\ dV$$

$$\Longrightarrow$$

For a linear bar element

$$\underline{n}^e = \begin{bmatrix} 1-\xi \\ \xi \end{bmatrix}$$

$$\underline{b}^{eT} = \frac{1}{L}\begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$[k^e] = \frac{EA}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

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